

Riemann-Liouville and higher dimensional Hardy operators for non-negative decreasing function in $L^{p(\cdot)}$ spaces

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Abstract. In this paper one-weight inequalities with general weights for Riemann-Liouville transform and n -dimensional fractional integral operator in variable exponent Lebesgue spaces defined on \mathbb{R}^n are investigated. In particular, we derive necessary and sufficient conditions governing one-weight inequalities for these operators on the cone of non-negative decreasing functions in $L^{p(x)}$ spaces.

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1. INTRODUCTION

We derive necessary and sufficient conditions governing the one-weight inequality for the Riemann-Liouville operator

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad 0 < \alpha < 1,$$

and n -dimensional fractional integral operator

$$I_\alpha g(x) = \frac{1}{|x|^\alpha} \int_{|y| < |x|} \frac{g(t)}{|x-t|^{n-\alpha}} dt \quad 0 < \alpha < n,$$

on the cone of non-negative decreasing function in $L^{p(x)}$ spaces.

In the last two decades a considerable interest of researchers was attracted to the investigation of the mapping properties of integral operators in so called Nakano spaces $L^{p(\cdot)}$ (see e.g., the monographs [5], [7] and references therein). Mathematical problems related to these spaces arise in applications to mechanics of the continuum medium. For example, M. Ružička [19] studied the problems in the so called rheological and electrorheological fluids, which lead to spaces with variable exponent.

Weighted estimates for the Hardy transform

$$(Hf)(x) = \int_0^x f(t) dt, \quad x > 0,$$

in $L^{p(\cdot)}$ spaces were derived in the papers [8] for power-type weights and in [11], [12], [15], [6], [17] for general weights. The Hardy inequality for non-negative decreasing functions was studied in [3], [4].

Weighted problems for the Riemann-Liouville transform in $L^{p(x)}$ spaces were explored in the papers [10], [11], [2], [14] (see also the monograph [18]).

Historically, one and two weight Hardy inequalities on the cone of non-negative decreasing functions defined on \mathbb{R}_+ in the classical Lebesgue spaces were characterized by M. A. Arino and B. Muckenhoupt [1] and E. Sawyer [22] respectively.

It should be emphasized that the operator $I_\alpha f(x)$ is the weighted truncated potential. The trace inequity for this operator in the classical Lebesgue spaces was established by E. Sawyer [21] (see also the monograph [13], Ch.6 for related topics).

In general, the modular inequality

$$\int_0^1 \left| \int_0^x f(t) dt \right|^{q(x)} v(x) dx \leq c \int_0^1 |f(t)|^{p(t)} w(t) dt \quad (*)$$

for the Hardy operator is not valid (see [23], Corollary 2.3, for details). Namely the following fact holds: if there exists a positive constant c such that inequality (*) is true for all $f \geq 0$, where q ; p ; w and v are non-negative measurable functions, then there exists $b \in [0, 1]$ such that $w(t) > 0$ for almost every $t < b$; $v(x) = 0$ for almost every $x > b$, and $p(t)$ and $q(x)$ take the same constant values almost everywhere for $t \in (0; b)$ and $x \in (0; b) \cap \{v \neq 0\}$.

To get the main result we use the following pointwise inequities

$$c_1(Tf)(x) \leq (R_\alpha f)(x) \leq c_2(Tf)(x),$$

$$c_3(Hg)(x) \leq (I_\alpha g)(x) \leq c_4(Hg)(x),$$

for non-negative decreasing functions, where c_1 , c_2 , c_3 and c_4 are constants are independents of f , g and x , and

$$Tf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad Hg(x) = \frac{1}{|x|^n} \int_{|y| < |x|} g(y) dy.$$

In the sequel by the symbol $Tf \approx Tg$ we means that there are positive constants c_1 and c_2 such that $c_1 Tf(x) \leq Tg(x) \leq c_2 Tf(x)$. Constants in inequalities will be mainly denoted by c or C ; the symbol \mathbb{R}_+ means the interval $(0, +\infty)$.

2. PRELIMINARIES

We say that a radial function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is decreasing if there is a decreasing function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $g(|x|) = f(x)$, $x \in \mathbb{R}^n$. We will denote g

again by f . Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a measurable function, satisfying the conditions $p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) > 0$, $p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) < \infty$.

Given $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and a non-negative measurable function (weight) u in \mathbb{R}^n , let us define the following local oscillation of p :

$$\varphi_{p(\cdot), u(\delta)} = \operatorname{ess\,sup}_{x \in B(0, \delta) \cap \operatorname{supp} u} p(x) - \operatorname{ess\,inf}_{x \in B(0, \delta) \cap \operatorname{supp} u} p(x),$$

where $B(0, \delta)$ is the ball with center 0 and radius δ .

We observe that $\varphi_{p(\cdot), u(\delta)}$ is non-decreasing and positive function such that

$$\lim_{\delta \rightarrow \infty} \varphi_{p(\cdot), u(\delta)} = p_u^+ - p_u^-, \quad (1)$$

where p_u^+ and p_u^- denote the essential infimum and supremum of p on the support of u , respectively.

By the similar manner it is defined (see [3]) the function $\psi_{p(\cdot), u(\eta)}$ for an exponent $p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and weight v on \mathbb{R}_+ :

$$\varphi_{p(\cdot), v(\varepsilon)} = \operatorname{ess\,sup}_{x \in B(0, \varepsilon) \cap \operatorname{supp} v} p(x) - \operatorname{ess\,inf}_{x \in B(0, \varepsilon) \cap \operatorname{supp} v} p(x),$$

Let $D(\mathbb{R}_+)$ be the class of non-negative decreasing functions on \mathbb{R}_+ and let $DR(\mathbb{R}^n)$ be the class of all non-negative radially decreasing functions on \mathbb{R}^n . Suppose that u is measurable a.e. positive function (weight) on \mathbb{R}^n . We denote by $L^{p(x)}(u, \mathbb{R}^n)$, the class of all non-negative functions on \mathbb{R}^n for which

$$S_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} u(x) d\mu(x) < \infty.$$

For essential properties of $L^{p(x)}$ spaces we refer to the papers [16] [20] and the monographs [7], [5].

Under the symbol $L_{dec}^{p(x)}(u, \mathbb{R}_+)$ we mean the class of non-negative decreasing functions on \mathbb{R}_+ from $L^{p(x)}(u, \mathbb{R}^n) \cap DR(\mathbb{R}^n)$.

Now we list the well-known results regarding one-weight inequality for the operator T . For the following statement we refer to [1].

Theorem A. *Let r be constant such that $0 < r < \infty$. Then the inequality*

$$\int_0^\infty v(x) (Tf(x))^r dx \leq c \int_0^\infty v(x) (f(x))^r dx, \quad f \in L^r(v, \mathbb{R}_+), \quad f \downarrow \quad (2)$$

for a weight v holds, if and only if there exists a positive constant C such that for all $s > 0$

$$\int_s^\infty \left(\frac{s}{x}\right)^r v(x) dx \leq C \int_0^s v(x) dx. \quad (3)$$

Condition (3) is called B_r condition and was introduced in [1].

Theorem B[3]. *Let v be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\psi_{p(\cdot), v(0+)} = 0$. The following facts are equivalent:*

(a) *There exists a positive constant c such that for any $f \in D(\mathbb{R}_+)$,*

$$\int_0^\infty (Tf(x))^{p(x)} v(x) dx \leq C \int_0^\infty (f(x))^{p(x)} v(x) dx. \quad (4)$$

(b) *For any $r, s > 0$,*

$$\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} v(x) dx \leq C \int_0^r \frac{v(x)}{s^{p(x)}} dx. \quad (5)$$

(c) $p|_{\text{supp } v} \equiv p_0$ a.e and $v \in B_{p_0}$.

Proposition 2.1. *For the operators T, H, R_α and I_α , the following relations hold:*

(a)

$$R_\alpha f \approx Tf, \quad 0 < \alpha < 1, \quad f \in D(\mathbb{R}_+);$$

(b)

$$I_\alpha g \approx Hg, \quad 0 < \alpha < n, \quad g \in DR(\mathbb{R}^n).$$

Proof. (a) Upper estimate. Represent $R_\alpha f$ as follows:

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}} dt + \frac{1}{x^\alpha} \int_{x/2}^x \frac{f(t)}{(x-t)^{1-\alpha}} dt = S_1(x) + S_2(x).$$

Observe that if $t < x/2$, then $x/2 < x-t$. Hence

$$S_1(x) \leq c \frac{1}{x} \int_0^{x/2} f(t) dt \leq cTf(x),$$

where the positive constant c does not depend on f and x . Using the fact that f is decreasing we find that

$$S_2(x) \leq cf(x/2) \leq cTf(x).$$

Lower estimate follows immediately by using the fact that f is non-negative and the obvious estimate $x - t \leq x$ and $0 < t < x$.

(b) Upper estimate. Let us represent the operator I_α as follows:

$$\begin{aligned} I_\alpha g(x) &= \frac{1}{|x|^\alpha} \int_{|y| < |x|/2} \frac{g(y)}{|x-y|^{n-\alpha}} dy + \frac{1}{|x|^\alpha} \int_{|x|/2 < |y| < |x|} \frac{g(y)}{|x-y|^{n-\alpha}} dy \\ &=: S'_1(x) + S'_2(x). \end{aligned}$$

Since $|x|/2 \leq |x-y|$ for $|y| < |x|/2$ we have that

$$S'_1(x) \leq \frac{c}{|x|^n} \int_{|y| < |x|/2} g(y) dy \leq cH g(x).$$

Taking into account the fact that f is radially decreasing on \mathbb{R}^n we find that there is a decreasing function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$S'_2(x) \leq f(|x|/2) \cdot \frac{1}{|x|^\alpha} \int_{|x|/2 < |y| < |x|} |x-y|^{\alpha-n} dy$$

Let $F_x = \{y : |x|/2 < |y| < |x|\}$. Then we have

$$\begin{aligned} \int_{F_x} |x-y|^{\alpha-n} dy &= \int_0^\infty |\{y \in F_x : |x-y|^{\alpha-n} > t\}| dt \\ &\leq \int_0^{|x|^{\alpha-n}} |\{y \in F_x : |x-y|^{\alpha-n} > t\}| dt + \int_{|x|^{\alpha-n}}^\infty |\{y \in F_x : |x-y|^{\alpha-n} > t\}| dt \\ &=: I_1 + I_2. \end{aligned}$$

It is easy to see that

$$I_1 \leq \int_0^{|x|^{\alpha-n}} |B(0, |x|)| dt = c|x|^\alpha,$$

while using the fact that $\frac{n}{n-\alpha} > 1$ we find that

$$I_2 \leq \int_{|x|^{\alpha-n}}^\infty |\{y \in F_x : |x-y| \leq t^{\frac{1}{\alpha-n}}\}| dt \leq c \int_{|x|^{\alpha-n}}^\infty t^{\frac{n}{\alpha-n}} dt = c_{\alpha,n} |x|^\alpha.$$

Finally we conclude that

$$S'_2(x) \leq cf(|x|/2) \leq cHf(x).$$

Lower estimate follows immediately by using the fact that f is non-negative and the obvious estimate $|x - y| \leq |x|$, where $0 < |y| < |x|$. \square

We will also need the following statement:

Lemma 2.2. *Let r be a constant such that $0 < r < \infty$. Then the inequality*

$$\int_{\mathbb{R}^n} (Hf(x))^r u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^r u(x) dx, \quad f \in L_{dec}^r(u, \mathbb{R}^n) \quad (6)$$

holds, if and only if there exists a positive constant C such that for all $s > 0$,

$$\int_{|x|>s} \left(\frac{s}{|x|}\right)^r |x|^{r(1-n)} u(x) dx \leq C \int_{|x|<s} |x|^{r(1-n)} u(x) dx. \quad (7)$$

Proof. We shall see that inequality (6) is equivalent to the inequality

$$\int_0^\infty \tilde{u}(t) (T\bar{f}(t))^r dt \leq C \int_0^\infty \tilde{u}(t) (\bar{f}(t))^r dt,$$

where $\tilde{u}(t) = t^{(n-1)(1-r)} \bar{u}(t)$, $\bar{f}(t) = t^{n-1} f(t)$ and $\bar{u}(t) = \int_{S_0} u(t\bar{x}) d\sigma(\bar{x})$.

Indeed, using polar the coordinates in \mathbb{R}^n we have

$$\begin{aligned} \int_{\mathbb{R}^n} (Hf(x))^r u(x) dx &= \int_{\mathbb{R}^n} u(x) \left(\frac{1}{|x|^n} \int_{|y|<|x|} f(y) dy \right)^r dx \\ &= \int_0^\infty t^{n-1} \left(\frac{1}{|t|^n} \int_{|y|<|x|} f(y) dy \right)^r \left(\int_{S_0} u(t\bar{x}) d\sigma \bar{x} \right) dt \\ &= C \int_0^\infty t^{n-1} t^{-nr} t^r \left(\frac{1}{t} \int_0^t \tau^{n-1} f(\tau) d\tau \right)^r \bar{u}(t) dt \\ &= C \int_0^\infty t^{n-1} t^{r(1-n)} \bar{u}(t) \left(\frac{1}{t} \int_0^t \bar{f}(\tau) d\tau \right)^r dt \\ &\leq C \int_0^\infty \tilde{u}(t) (\bar{f}(t))^r dt \\ &= C t^{(n-1)(1-r)} t^{(n-1)r} (f(t))^r dt \\ &= C \int_{\mathbb{R}^n} (f(x))^r u(x) dx. \end{aligned}$$

□

3. THE MAIN RESULTS

To formulate the main results we need to prove

Proposition 3.1. *Let u be a weight on \mathbb{R}^n and $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\varphi_{p(\cdot), u(0+)} = 0$. The following statements are equivalent:*

(a) *There exists a positive constant C such that for any $f \in DR(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (Hf(x))^{p(x)} u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x) dx. \quad (8)$$

(b) *For any $r, s > 0$,*

$$\int_{|x|>r} \left(\frac{r}{s|x|^n} \right)^{p_0} u(x) dx \leq C \int_{B(0,r)} \frac{|x|^{(1-n)p_0} u(x)}{s^{p_0}} dx. \quad (9)$$

(c) $p|_{\text{supp } u} \equiv p_0$ a.e and $u \in B_{p_0}$.

Proof. We use the arguments of [3]. To show that (a) implies (b) it is enough to test the modular inequality (8) for the function $f_{r,s}(x) = \frac{1}{s} \chi_{B(0,r)}(x) |x|^{1-n}$, $s, r > 0$. Indeed, it can be checked that

$$Hf_{r,s}(x) = \begin{cases} \frac{1}{|x|^n s} \int_{|y| \leq |x|} |y|^{1-n} dy, & \text{if } |x| \leq r; \\ \frac{1}{|x|^n s} \int_{|y| \leq r} |y|^{1-n} dy, & \text{if } |x| > r \end{cases}.$$

Further, we find that

$$\int_{|x|>r} u(x) (Hf_{r,s})^{p(x)} dx \leq \int_{\mathbb{R}^n} u(x) (Hf_{r,s})^{p(x)} dx \leq C \int_{\mathbb{R}^n} u(x) \left(\frac{1}{s} \chi_{B(0,r)}(x) |x|^{1-n} \right)^{p(x)} dx.$$

Therefore

$$\int_{|x|>r} u(x) \left(\frac{r}{s|x|^n} \right)^{p(x)} dx \leq C \int_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx.$$

To obtain (c) from (b) we are going to prove that condition (b) implies that $\varphi_{p(\cdot), u(\delta)}$ is a constant function, namely $\varphi_{p(\cdot), u(\delta)} = p_u^+ - p_u^-$ for all $\delta > 0$. This fact and the hypothesis on $\varphi_{p(\cdot), u(\delta)}$ implies that $\varphi_{p(\cdot), u(\delta)} \equiv 0$, and hence due to (1),

$$p|_{\text{supp } u} \equiv p_u^+ - p_u^- \equiv p_0 \quad \text{a.e..}$$

Finally (9) means that $u \in B_{p_0}$. Let us suppose that $\varphi_{p(\cdot),u}$ is not constant. Then one of the following conditions hold:

(i) there exists $\delta > 0$ such that

$$\alpha = \operatorname{ess\,sup}_{x \in B(0,\delta) \cap \operatorname{supp} u} p(x) < p_u^+ < \infty, \quad (10)$$

and hence, there exists $\epsilon > 0$ such that

$$|\{ |x| > \delta : p(x) \geq \alpha + \epsilon \} \cap \operatorname{supp} u| > 0,$$

or

(ii) there exists $\delta > 0$ such that

$$\beta = \operatorname{ess\,inf}_{x \in B(0,\delta) \cap \operatorname{supp} u} p(x) > p_u^- > 0, \quad (11)$$

and then, for some $\epsilon > 0$,

$$|\{ |x| > \delta : p(x) \leq \beta - \epsilon \} \cap \operatorname{supp} u| > 0.$$

In the case (i) we observe that condition (b) for $r = \delta$, implies that

$$\int_{|x| > \delta} \left(\frac{\delta}{s} \right)^{p(x)} \frac{u(x)}{|x|^{np(x)}} dx \leq C \int_{B(0,\delta)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx.$$

Then using (10) we obtain, for $s < \min(1, \delta)$,

$$\left(\frac{\delta}{s} \right)^{\alpha + \epsilon} \int_{\{|x| \geq \delta : p(x) \geq \alpha + \epsilon\}} \frac{u(x)}{|x|^{np(x)}} dx \leq \frac{C}{s^\alpha} \int_{B(0,\delta)} u(x) |x|^{(1-n)p(x)} dx,$$

which is clearly a contradiction if we let $s \downarrow 0$. Similarly in the case (ii) let us consider the same condition (b) for $r = \delta$, and fix now $s > 1$. Taking into account (11) we find that:

$$\frac{1}{s^{\beta - \epsilon}} \int_{\{|x| \geq \delta : p(x) \leq \beta - \epsilon\}} \left(\frac{\delta}{|x|^n} \right)^{p(x)} u(x) dx \leq \frac{C}{s^\beta} \int_{B(0,\delta)} |x|^{(1-n)p(x)} u(x) dx,$$

which is a contradiction if we let $s \uparrow \infty$.

Finally, the fact that condition (c) implies (a) follows from [1, Theorem 1.7] \square

Theorem 3.2. *Let u be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$. Assume that $\psi_{p(\cdot),v(0^+)} = 0$. The following facts are equivalent:*

(i) *There exists a positive constant C such that for any $f \in D(\mathbb{R}_+)$,*

$$\int_{\mathbb{R}_+} (R_\alpha f(x))^{p(x)} v(x) dx \leq C \int_{\mathbb{R}_+} (f(x))^{p(x)} v(x) dx.$$

(ii) *condition (5) holds;*

(iii) condition (c) of Theorem B is be satiesfied.

Proof. Proof follows by using Theorems B and Proposition 2.1(a). \square

Theorem 3.3. Let u be a weight on \mathbb{R}^n and $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\varphi_{p(\cdot), u(0^+)} = 0$. The following facts are equivalent:

(i) There exists a positive constant C such that for any $f \in DR(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (I_\alpha f(x))^{p(x)} u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x) dx.$$

(ii) condition (9) holds;

(iii) condition (c) of Proposition 3.1 holds.

Proof. Proof follows by using Propositions 3.1 and Proposition 2.1 (b). \square

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